Generating compact spaces from trees

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Outline

Strolling through the landscape

All trees considered are $\aleph_1$-trees, satisfying extra normality conditions:

1. for any $s \neq t$ in T of limit height (>0), $pred(s) \neq pred(t)$.
2. $\forall t \in T, succ(t)$ is uncountable,
3. $\forall t \in T, immsucc(t)$ is infinite,

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Every normal tree T gives rise to a (compact T_1) ordered space L. Moreover, there is an injective homomorphism from T into L.

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1. L is a linearly ordered set,
2. $a_t < b_t$ for $t \in T$,
3. $a_s < a_t$ and $b_t < b_s$ for $s < t$ in T,
4. $b_t < a_s$ or $b_s < a_t$ for $s, t \in (T\})$ incomparable.

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Proof continued.

If $\mathcal{A} = (L, <, c : c \in \mathcal{L})$ is a model of Σ, it is easy to see that there is an order-preserving injection from T into L mapping t to a_t.

What remains to show then is that Σ has a model. Let Δ be a finite subset of Σ, and let T_0 be the finite subset of T consisting of exactly those points of T to which sentences in Δ refer. It is very simple to do the rest on fingers, ensuring that conditions (1-4) above are met. Therefore Δ does indeed have a model. By the compactness theorem, Σ has a model. □

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Wallman's Representation Theorem

Stone's representation theorem

Duality between the category of Boolean algebras and the category of Stone spaces.

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Definitions

A lattice is *distributive* if it models the sentences

$$a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c),$$

$$a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c).$$

A lattice is *disjunctive* if it models the sentence

$$\forall a, b \exists c [a \not\leq b \to c \neq 0 \wedge c \leq a \wedge c \sqcap b = 0].$$

- Each space X comes with a lattice $\mathcal{K}(X)$, its family of closed sets, with $\cap$ and $\cup$ as the operations. The lattice is distributive, disjunctive, and bounded (with $\emptyset$ and X).
- On the other hand, for each distributive disjunctive lattice L, there is a compact T_1-space X with a base for its closed sets that is an isomorphic image of L.

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The space X is obtained in the usual way (Stone topology). Take the set of ultrafilters wL of L with the topology generated by $\{A* : A \in L\}$, where $A*$ is the set of ultrafilters on L that contain the element A of L.

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What remains then is to show that Σ has a model. Again, let Δ be a finite subset of Σ, and let T_0 be the finite subset of T consisting of exactly those points of T to which sentences in Δ refer. Obviously, we can continue, calculating on fingers. As a consequence, conditions (1-3) above are satisfied, and so Δ has a model. By the compactness theorem, Σ has a model.

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Consequences

Embedding trees in hyperspaces

Every normal tree T gives rise to a (compact T_1) ordered space X. Moreover, T embeds into the hyperspace $\mathcal{K}(X)$ of closed subsets of X.

Remarks

1. There are no existential sentences in the theory, apart from disjunctiveness. Therefore in almost all situations, a substructure is a submodel. Therefore we may find a lot of compact spaces generated by a subbase for closed sets which is a tree under the reverse inclusion order (pseudo non-Archimedean?).

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2. In all cases, the lattices produced may have large cardinality (we have no control on that). Fortunately, the Lowenheim-Skolem theorem puts things in order. In particular, we may take an elementary sublattice of L containing $\{a_t : t \in T\}$, and of cardinality $\aleph_1$. That is, whatever the weight ($\geq \aleph_1$) of the space, we can obtain a quotient of weight $\aleph_1$ using the Lowenheim-Skolem theorem.

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3 We may assume that all elements of $\{a_t : t \in T\}$ are connected, so that T embeds in the hyperspace $C(X)$ of all subcontinua of X.

4 In general, one may add any first-order property of topological spaces, expressed in terms of closed sets. One has to be careful: Every finite subset of the theory has to be satisfiable. In addition, if one puts introduces existential operators in sentences, one may lose special properties of (sub)bases.

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Summary of one approach: Start with a normal tree. Embed the tree in a lattice, which is a base for closed sets of a compact space. Restrict to constant elements, and close under the functions (cap,cup). Viola. The tree itself is a subbase for closed sets of the obtained space.

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Examples

Natural examples of non-metric results include the Aronszajn compacta of K. Kunen and J. Hart.

Definition

An *embedded Aronszajn compactum* is a closed subspace $X \subseteq [0,1]^{\omega_1}$ with $w(X) = \aleph_1$ and $\chi(X) = \aleph_0$ such that for some club $C \subseteq \omega_1$: for each $\alpha \in C, \mathcal{L}_\alpha := \{x \in X_\alpha : |(\sigma_\alpha^{\omega_1})^{-1}\{x\}| > 1\}$ is countable.

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For each such X, define $T = T(X) := \bigcup\{L_\alpha : \alpha \in C\}$, and let $\lhd$ denote the following order: if $\alpha, \beta \in C, \alpha < \beta, x \in \mathcal{L}_\alpha$ and $y \in \mathcal{L}_\beta$ then $x \lhd y$ iff $x = \pi_\alpha^\beta(y)$. $(T, \lhd)$ is then an $\aleph_1$-tree.

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Kunen, Hart

Assume $\diamondsuit$. For each of the following 2x3 = 6 possibilities, there is an HS, HL, Aronszajn compactum X with tree $T = T(X)$.
Possibilities for T:

a. T is Suslin.

b. T is special.

Possibilities for X:

α. $dim(X) = 0$.

β. $dim(X) = 1$ and X is connected and locally connected.

γ. $dim(X) = \infty$ and X is connected and locally connected.

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Daniel et al.

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$$< U >= \{F \in 2^X : F \subseteq U\}$$

and

$$> U <= \{F \in 2^X : F \cap U \neq \emptyset\}.$$

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What about $C(X)$?

Applications

Let X be a compactum. $\mathcal{K}(X)$ and $C(X)$, apart from being topological spaces, are partially ordered sets. We always fix the order $\leq_X$ to be reverse inclusion.

What can the order structure of those hyperspaces tell us about the topology?

An immediate result, based on the Wallman representation theorem and a work of Wolk: for a compactum X, the structure of $\mathcal{K}(X)$, considered as a partially ordered set, determines X up to homeomorphism.

What about $C(X)$?

A compactum X is *hereditarily indecomposable* if for every two intersecting subcontinua of X one is contained in the other. This means that $(C(X), \leq_X)$ is a pseudo-tree!

Nikiel

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